JOURNAL OF APPROXIMATION THEORY 37, 356–371 (1983)

On H-Sets in Bivariate Rational Approximation

J. P. THIRAN, P. DEFERT, AND E. PANIER

Department of Mathematics, Facultés Universitaires de Namur, B-5000 Namur, Belgium

Communicated by G. Meinardus

Received April 15, 1982

1. INTRODUCTION AND NOTATIONS

In this paper, we consider the following generalized rational approximation problem. Let $C(B)$ be the Banach space of continuous real-valued functions on some compact subset B of the Euclidean plane $\mathbb{R}^2$, which is equipped with the Chebyshev norm $\|f\| = \sup\{|f(t)|;\ t = (x, y) \in B\}$ for all $f \in C(B)$. If U and V denote two finite-dimensional subspaces of $C(B)$, the approximating family is given by

$$G = \{g = u/v;\ u \in U,\ v \in V \text{ with } v(t) > 0 \text{ for all } t \in B\}. \tag{1}$$

When G is nonempty, the problem is to approximate $f \in C(B)\backslash G$ by elements of G to reach the minimal distance $\rho(f, G) = \inf\{\|f - g\|;\ g \in G\}$.

For general nonlinear families, Collatz [5] defined an H-set as a subset M of B which verifies hypothesis H: there is a partition $M_1 \cup M_2$ of M such that no pair $g_1, g_2 \in G$ satisfies $g_1 - g_2 > 0$ in M_1 and $g_1 - g_2 < 0$ in M_2. An H-set is said to be minimal if it does not include any proper subset which is an H-set. The notion of H-set is important for obtaining bounds on the minimal distance, through the inclusion theorem: given $\hat{g} \in G$ whose error $f - \hat{g}$ is positive on M_1 and negative on M_2, or vice versa, the following inclusion holds:

$$\inf_{t \in M} |f(t) - \hat{g}(t)| \leqslant \rho(f, G) \leqslant \|f - \hat{g}\|. \tag{2}$$

For testing hypothesis H by the family (1), the difference of any two elements $g_1 = u_1/v_1$ and $g_2 = u_2/v_2$ of G has the same sign in B as $u_1 v_2 - u_2 v_1$, due to the fact that both denominators are positive. Therefore, when U and V are spaces of two-variable polynomials of total degrees μ and ν, respectively,

0021-9045/83 $3.00

$$U = P_\mu = \left\{ \sum_{i+j \leqslant \mu} a_{ij} x^i y^j;\, a_{ij} \in \mathbb{R} \right\}, \qquad \dim U = (\mu+1)(\mu+2)/2,$$

$$V = P_\nu = \left\{ \sum_{i+j \leqslant \nu} b_{ij} x^i y^j;\, b_{ij} \in \mathbb{R} \right\}, \qquad \dim V = (\nu+1)(\nu+2)/2,$$

Collatz [5, 6] identifies H-sets with those for linear approximation by the space $P_{\mu+\nu}$, which have been investigated in several papers (see, for instance, [1, 5, 6, 8, 11, 13–16]).

If the bounds coincide in (2), $\hat{g}$ is a minimal solution. Whence, in linear approximation problems, H-sets play a central role in characterization theorems. On the contrary, in the rational case, one gets only a sufficient optimality condition, because the definition of H-set is too general for purpose of proving the inclusion theorem. In fact, one has to check hypothesis H only for $\hat{g} - g$, where $\hat{g}$ is the given element of G while g is arbitrary. This observation motivates a more restricted definition of H-set

DEFINITION 1. *A subset $M = M_1 \cup M_2$ of B with $M_1 \cap M_2 = \varnothing$, is an H-set relative to a fixed $\hat{g} \in G$ if there is no $g \in G$ which satisfies $\hat{g} - g > 0$ in M_1 and $\hat{g} - g < 0$ in M_2.*

For rational functions (1), the definition amounts to saying that no element of the linear space $W = \hat{v}U + \hat{u}V$ is positive in M_1 and negative in M_2 or vice versa. By standard arguments [8], one gets a dual equivalent definition: $M = \{t_1, t_2, \ldots, t_{m+1}\} \subset B$ is an H-set relative to $\hat{g}$ if one has

$$\sum_{i=1}^{m+1} \lambda(t_i)\, w(t_i) = 0, \qquad \sum_i |\lambda(t_i)| > 0, \qquad w \in W = \hat{v}U + \hat{u}V. \tag{3}$$

In this setting, H-sets are equivalent to extremal signatures introduced in [2] for rational approximation. With this more accurate definition of H-sets, identical lower and upper bounds in (2) yield the Kolmogorov characterization theorem [4, p. 159].

KOLMOGOROV THEOREM. *The element $\hat{g} \in G$ is a best approximation of f iff no $w \in W = \hat{v}U + \hat{u}V$ has the same sign as $f - \hat{g}$ on the extremal point set $E(\hat{g}) = \{t \in B;\, |f(t) - \hat{g}(t)| = \|f - \hat{g}\|\}$.*

We shall thus investigate H-sets for linear approximation by the space $W = \hat{v}U + \hat{u}V$, where $U = P_\mu$ and $V = P_\nu$ while $\hat{u}$ and $\hat{v}$ are fixed polynomials of degrees at most equal to μ and ν, respectively.

2. Dimension of the Space W

We assume that $\hat{g}$ is expressed in an irreducible form, i.e., $\hat{u}$ and $\hat{v}$ do not have a common factor. If $\hat{g}=0$, we adopt the convention $\hat{u}=0$, $\hat{v}=1$ so that $W=P_\mu$ and all H-sets correspond to the classical linear theory. For $\hat{g}\neq 0$, we define the defect of $\hat{g}$ by $\delta=\min\{\mu-\hat{\mu}, \nu-\hat{\nu}\}$, where $\hat{\mu}$ and $\hat{\nu}$ denote the actual degrees of $\hat{u}$ and $\hat{v}$. Hence, we have $W\subseteq P_{\mu+\nu-\delta}$. With the kind of argument used in the univariate case [4, p. 162], we prove

THEOREM 1. *For $\hat{g}\neq 0$, the dimension of $W=\hat{v}U+\hat{u}V$ is given by*

$$\dim W=\dim P_{\mu+\nu-\delta}-(\mu-\delta)(\nu-\delta). \tag{4}$$

By the modular law for the sum of spaces, we get $\dim W=\dim(\hat{v}U)+\dim(\hat{u}V)-\dim\{(\hat{v}U)\cap(\hat{u}V)\}$ in which $\dim(\hat{v}U)=\dim P_\mu$ and $\dim(\hat{u}V)=\dim P_\nu$. On the other hand, any element w of $(\hat{v}U)\cap(\hat{u}V)$ satisfies $w=\hat{v}u=\hat{u}v$ with $u\in U$ and $v\in V$. As $\hat{u}$ and $\hat{v}$ are prime polynomials, $\hat{v}$ divides v, i.e., $v=q\hat{v}$, which implies $u=q\hat{u}$. Whence, the degree of the polynomial q equals the defect δ and we have

$$\dim W=\dim P_\mu+\dim P_\nu-\dim P_\delta \tag{5}$$

or $\dim W=(\mu+1)(\mu+2)/2+(\nu+1)(\nu+2)/2-(\delta+1)(\delta+2)/2$. By some algebraic manipulations, this last expression can be rewritten as $\dim W=(\mu+\nu-\delta+1)(\mu+\nu-\delta+2/2-(\mu-\delta)(\nu-\delta)$. ∎

Actually, result (5) does not depend on the number of variables. If we apply it to univariate polynomials which are characterized by $\dim P_n=n+1$, we get $\dim W=\dim P_{\mu+\nu-\delta}$ and $W=P_{\mu+\nu-\delta}$. Therefore, any minimal H-set consists of $\mu+\nu-\delta+2$ points with alternating signs. In this way, we rediscover the classical alternation property of univariate rational approximation [4, p. 163].

On the contrary, for two-variable functions, if $(\mu-\delta)(\nu-\delta)$ is nonzero, W is a proper subset of $P_{\mu+\nu-\delta}$. This peculiarity can be explained by a fundamental theorem of algebraic geometry, which is due to Noether.

3. Noether Theorem

The Noether theorem [9] provides necessary and sufficient conditions for a bivariate polynomial to belong to the space $W=\hat{v}U+\hat{u}V$. They concern the behaviour of w at each intersection point of the two algebraic plane curves defined by $\hat{u}(x,y)=0$ and $\hat{v}(x,y)=0$.

NOETHER THEOREM. *A polynomial w can be written as $\hat{v}u + \hat{u}v$ in which $\hat{u}$ and $\hat{v}$ denote given polynomials having no common factors, while u and v are arbitrary iff the Noether condition is fulfilled at each intersection point $q_i = (x_i, y_i)$ of the two basic curves $\hat{u} = 0$ and $\hat{v} = 0$: there exists a pair of polynomials u_i, v_i such that the difference $w - \hat{v}u_i - \hat{u}v_i$ expanded in powers of $x - x_i$, $y - y_i$, starts with terms of degree at least equal to a number ρ_i called the Noether exponent at q_i.*

Since the curves $\hat{u} = 0$ and $\hat{v} = 0$ have no common part, due to Bezout theorem [7, p. 10], they have exactly $(\mu - \delta)(\nu - \delta)$ intersection points provided all points at infinity are taken into account. For instance, if $\mu = 1$ and $\nu = 2$, the rational function $\hat{g} = 1/(x^2 - y^2 + x)$ has a zero defect and we compute the two intersection points by expressing $\hat{g}$ as a quotient of two homogeneous polynomials $Z/(X^2 - Y^2 + XZ)$: this gives the points at infinity $X_1 = 1$, $Y_1 = 1$, $Z_1 = 0$ and $X_2 = 1$, $Y_2 = -1$, $Z_2 = 0$. By (4), all conditions arising from the $(\mu - \delta)(\nu - \delta)$ intersection points are independent.

The Noether exponent can be characterized in terms of geometrical properties of the two curves $\hat{u} = 0$ and $\hat{v} = 0$ at their intersection points [9]. For the sake of simplicity, we consider an intersection point at the origin, i.e., $q = (0, 0)$, which is of order σ on $\hat{u} = 0$ and of order τ on $\hat{v} = 0$

$$\hat{u}(x, y) = \sum_{i \geqslant \sigma} \hat{u}_i(x, y), \qquad \hat{v}(x, y) = \sum_{i \geqslant \tau} \hat{v}_i(x, y),$$

where $\hat{u}_i$ and $\hat{v}_i$ are homogeneous polynomials of degree i in x and y. The first nonzero polynomials $\hat{u}_\sigma$ and $\hat{v}_\tau$ define the various tangents at the origin

$$\hat{u}_\sigma(1, \alpha) = K_u \prod_{i=1}^{\sigma} (\alpha - \alpha_i), \qquad \hat{v}_\tau(1, \beta) = K_v \prod_{i=1}^{\tau} (\beta - \beta_i).$$

The multiplicity κ of the intersection point q is at least $\sigma\tau$. Equality occurs iff the two curves have no contact at q, i.e., $\alpha_i \neq \beta_j$ for $i = 1, 2, \ldots, \sigma$ and $j = 1, 2, \ldots, \tau$. We can then state

PROPOSITION 1 ([9]). *The Noether exponent ρ is bounded as*

$$\rho \leqslant \kappa - (\sigma - 1)(\tau - 1), \tag{6}$$

and it reaches its upper bound iff the curves $\hat{u} = 0$ and $\hat{v} = 0$ have at most one common tangent which is simple for at least one of them.

Two particular examples where equality holds in (6) will be examined in the next sections. First, if q is an ordinary point on $\hat{u} = 0$, i.e., $\sigma = 1$, one has $\rho = \kappa$. Second, if $\hat{u} = 0$ and $\hat{v} = 0$ have no contact at q, one gets $\kappa = \sigma\tau$ and

$\rho = \sigma + \tau - 1$. For the general case of singular points, i.e., σ, $\tau > 1$, with some common tangents, a few partial results are known [9] but a complete treatment has not been given.

4. Ordinary Intersection Points

As pointed out in the last section, if the intersection point is ordinary on one basic curve, its multiplicity is identical to the Noether exponent. In that case, the Noether condition can take a more geometrical form [12].

THEOREM 2. *When $\hat{u} = 0$ and $\hat{v} = 0$ contain an intersection point q of multiplicity κ, which is ordinary on $\hat{u} = 0$, the polynomial w satisfies the Noether condition at q iff the curves $\hat{u} = 0$ and $w = 0$ have at q an intersection point of multiplicity at least equal to κ.*

For simple intersection points q, we have $\kappa = \sigma = \tau = 1$ and the Noether condition only requires that the curve passes through q. Whence, if we consider a rational function $\hat{g} = \hat{u}/\hat{v}$, having zero defect with no loss of generality, such that $\hat{u} = 0$ and $\hat{v} = 0$ intersect only in points of multiplicity one, investigating H-sets for W amounts to characterizing H-sets for polynomials of degree $\mu + \nu$, which vanish at $\mu\nu$ distinct points, i.e., $W = \{w \in P_r;\ w(q_j) = 0 \text{ for } j = 1, 2, \ldots, l\}$, where we set $\mu + \nu = r$ and $\mu\nu = l$ for convenience. The zeros q_j are all located outside B since $\hat{v}$ is positive in B.

In order to solve such linear approximation problems with side interpolation conditions, we can exploit the notion of support which was introduced by Carasso and Laurent [3] in connection with a generalized exchange algorithm. To this end, we denote by $\phi(t) = [1 x \cdots y^r]^{\mathrm{T}} \in \mathbb{R}^n$, where $n = (r+1)(r+2)/2$, the characteristic vector associated with P_r, and by $I = \operatorname{span}\{\phi(q_1), \phi(q_2), \ldots, \phi(q_l)\}$ with $\dim I = l$, the space coresponding to the interpolation conditions.

DEFINITION 2. *A set $S = \{t_1, t_2, \ldots, t_{m+1}\}$ is termed a support of I if there exist coefficients $\lambda(t_i)$, not all zero, such that $\sum_{i=1}^{m+1} \lambda(t_i)\, \phi(t_i) \in I$ or, equivalently,*

$$\sum_{i=1}^{m+1} \lambda(t_i)\, \phi(t_i) = \sum_{j=1}^{l} \mu(q_j)\, \phi(q_j), \qquad \sum_i |\lambda(t_i)| > 0. \tag{7}$$

The support is minimal when $S \backslash \{t_j\}$ is not a support of I for $j = 1, 2, \ldots, m+1$.

One readily obtains [3] a characterization of minimal supports and, at the same time, of minimal H-sets which, in view of (3), are supports of the null space.

PROPOSITION 2. *The support S is minimal iff all coefficients $\lambda(t_i)$ are nonzero and the dimension of the space spanned by $\phi(t_1),..., \phi(t_{m+1})$, $\phi(q_1),..., \phi(q_l)$, is $m+l$.*

The link between supports and H-sets relative to approximation problems with interpolation conditions is indicated in

THEOREM 3. *The set $S = \{t_1, t_2,..., t_{m+1}\}$ is an H-set for W iff it is a support of I for P_r.*

Proof. Sufficient condition. For a support S of I, we have (7) which can be written in the form

$$\sum_{i=1}^{m+1} \lambda(t_i)\, w(t_i) = \sum_{j=1}^{l} \mu(q_j)\, w(q_j), \qquad \sum_i |\lambda(t_i)| > 0, \quad w \in P_r.$$

As $w \in W \subseteq P_r$ satisfies $w(q_j) = 0$ for $j = 1, 2,..., l$, we get $\sum_{i=1}^{m+1} \lambda(t_i)\, w(t_i) = 0$ for all $w \in W$ and, by (3), S is an H-set for W.

Necessary condition. We first compute the characteristic vector associated with W. Any $w = \phi^{\mathrm{T}} a \in P_r$ with $a \in \mathbb{R}^n$ belongs to W if it vanishes at q_1, q_2,... q_l or, equivalently, if $La = 0$, where the matrix $L = [\phi(q_1)\, \phi(q_2) \cdots \phi(q_l)]^{\mathrm{T}}$ has rank l. Assuming its last l columns to be independent and partitioning L, a and ϕ accordingly,

$$L = [L_{n-l} L_l], \qquad a^{\mathrm{T}} = [a_{n-l}^{\mathrm{T}} a_l^{\mathrm{T}}], \qquad \phi^{\mathrm{T}} = [\phi_{n-l}^{T} \phi_l^{\mathrm{T}}],$$

we get $a_l = -L_l^{-1} L_{n-l} a_{n-l}$ so that any $w \in W$ is given by $w = \psi^T a_{n-l}$ in which ψ is the characteristic vector relative to W.

$$\psi(t) = \phi_{n-l}(t) - L_{n-l}^{\mathrm{T}} (L_l^{\mathrm{T}})^{-1} \phi_l(t).$$

From the identity

$$0 = \phi_l(t) - L_l^{\mathrm{T}} (L_l^{\mathrm{T}})^{-1} \phi_l(t),$$

the n-dimensional vector $\chi^{\mathrm{T}} = [\psi^{\mathrm{T}} 0]$ is given by

$$\chi(t) = \phi(t) - \sum_{j=1}^{l} \phi(q_j)\, x_j(t), \tag{8}$$

where $[x_1(t)\, x_2(t) \cdots x_l(t)]^{\mathrm{T}} = (L_l^{\mathrm{T}})^{-1} \phi_l(t)$.

Now, if S is an H-set for W, we have $\sum_{i=1}^{m+1} \lambda(t_i)\, \psi(t_i) = 0$ or $\sum_{i=1}^{m+1} \lambda(t_i)\, \chi(t_i) = 0$ with $\sum |\lambda(t_i)| > 0$. Using (8), we get (7) in which

$$\mu(q_j) = \sum_{i=1}^{m+1} \lambda(t_i)\, x_j(t_i). \quad \blacksquare$$

We easily deduce a more precise statement about minimal H-sets and minimal supports.

COROLLARY 1. *The set S is a minimal H-set for W iff it is a minimal support of I for P_r.*

We can then characterize minimal supports as

THEOREM 4. *The set $S = \{t_1, t_2,..., t_{m+1}\}$ is a minimal support of I for P_r iff there is an index h with $0 \leqslant h \leqslant l$ such that two conditions are fulfilled:*

(1) *$\{t_1,..., t_{m+1}\} \cup \{q_1,..., q_h\}$ is a minimal H-set for P_r,*

(2) *$\{q_{h+1},..., q_l\}$ is not a support of the space spanned by $\phi(t_1),..., \phi(t_{m+1}), \phi(q_1),..., \phi(q_h)$, with respect to P_r.*

Proof. Necessary condition. In view of Proposition 2, for a minimal support $\{t_1, t_2,..., t_{m+1}\}$ of I, we can write (7) with all nonzero coefficients $\lambda(t_i)$. As the space spanned by $\phi(t_1),..., \phi(t_{m+1}), \phi(q_1),..., \phi(q_l)$, has dimension $m+l$, we can delete any vector $\phi(t_i)$ to get a set of $m+l$ independent vectors. In (7), we suppose, with no loss of generality, that $\mu(q_j) \neq 0$ for $0 \leqslant j \leqslant h$ and $\mu(q_j) = 0$ for $h < j \leqslant l$. In this way, $t_1,..., t_{m+1}, q_1,..., q_h$ form a minimal H-set relative to P_r. Further, since the coefficients of (7) are unique within a common factor, the coefficients of $\phi(q_{h+1}),..., \phi(q_l)$ are necessarily zero and one has condition (2).

Sufficient condition. Condition (1) implies

$$\sum_{i=1}^{m+1} \lambda(t_i)\, \phi(t_i) = \sum_{j=1}^{h} \mu(q_j)\, \phi(q_j), \tag{9}$$

where all coefficients $\lambda(t_i)$ and $\mu(q_j)$ are nonzero. On the other hand, the space spanned by $\phi(t_1),..., \phi(t_{m+1}), \phi(q_1),..., \phi(q_h)$ has dimension $m+h$ and any subset of $m+h$ vectors consists of independent vectors. We can identify (9) with (7) by letting $\mu(q_j) = 0$ for $j > h$. Moreover, the vectors $\phi(t_1),..., \phi(t_m), \phi(q_1),..., \phi(q_l)$ are independent. Otherwise, the relationship

$$\sum_{i=1}^{m} \tilde{\lambda}(t_i)\, \phi(t_i) + \sum_{j=1}^{h} \tilde{\lambda}(q_j)\, \phi(q_j) + \sum_{j=h+1}^{l} \tilde{\mu}(q_j)\, \phi(q_j) = 0$$

TABLE I

r	$m+1$	C	D	Q
2	4	C_1	—	C_1
	6	C_2	—	C_2
	7	—	—	$\mathbb{R}^2$
3	5	C_1	—	C_1
	8	C_2	—	C_2
	9	—	$C_3^{(1)} \cap C_3^{(2)}$	$C_3^{(1)} \cap C_3^{(2)}$
	10	C_3	—	C_3
	11	—	—	$\mathbb{R}^2$

can be satisfied with at least one nonzero $\tilde{\mu}(q_j)$ since the vectors involved in the first two sums are independent. This contradicts condition (2). ∎

By Theorem 4(1), we have to know minimal *H*-sets relative to P_r. In [16], a general classification of minimal *H*-sets with respect to bivariate polynomials is based on the following argument. If a minimal *H*-set is composed of $m+1$ points, m is at most equal to n. When $m < n$, the $m+1$ points lie on $n-m$ independent algebraic plane curves of order r, denoted by $C^{(1)}, C^{(2)}, \ldots, C^{(n-m)}$. The algebraic variety $\bigcap_{i=1}^{n-m} C^{(i)}$ is the set Q of points whose characteristic vector belongs to the space spanned by those evaluated at the points of the *H*-set. In general, the $n-m$ curves have a factorization $C\Gamma^{(i)}$, $i = 1, 2, \ldots, n-m$, in which C is their common part while $\Gamma^{(1)}, \ldots, \Gamma^{(n-m)}$ have no common factor. Hence, Q is the union of C and of a set D of isolated points located on the curves $\Gamma^{(i)}$. If $m = n$, the points of the minimal *H*-set do not lie on any curve of order r and the set Q is actually the whole plane $\mathbb{R}^2$. For example, all minimal *H*-sets relative to degrees 2 and 3 are listed in Table I, where C_k stands for a curve of order k.

As an illustration of the foregoing developments, we consider *H*-sets on the cubic $y = x^3$. On that curve, third degree polynomials are spanned by the nine basic functions x^i with $0 \leqslant i \neq 8 \leqslant 9$. In order to compute the determinant of $[\phi(x_1)\,\phi(x_2) \cdots \phi(x_9)]$, where ϕ is the corresponding characteristic vector, we set $y = x_9$ to transform the determinant into a ninth degree polynomial in y, which can be written in the form

$$\prod_{\substack{i>j \\ \neq 9}} (x_i - x_j) \prod_{i=1}^{8} (y - x_i)(y - \alpha).$$

Imposing that the polynomial has no term in y^8 yields $\alpha = -\sum_{i=1}^{8} x_i$ so that

$$\det[\phi(x_1)\,\phi(x_2) \cdots \phi(x_9)] = \prod_{i>j} (x_i - x_j) \sum_i x_i. \tag{10}$$

Consequently, taking $x_1 < x_2 < \cdots < x_{10}$ such that $\sum_{i \neq j} x_i \neq 0$ for all j, we get a minimal H-set composed of ten points, whose associated variety Q is the whole cubic $y = x^3$. From (10), we easily show that the ten points have alternating signs except for those enclosing the point defined by $\sum_{i=1}^{10} x_i$. On the contrary, for $\sum_{i=1}^{9} x_i = 0$, we get nine isolated points which are given by the full intersection of $y = x^3$ with another independent cubic: there is no other point in the variety Q. If we are interested only in knowing the isolated points, we can replace the computation of the determinant by a more direct approach. Let $p(x)$ be a ninth degree polynomial having no term in x^8, which vanishes at nine distinct points $x_1, x_2, \ldots, x_9$. It is given by $K \prod_{i=1}^{9} (x - x_i)$ and the interpolation conditions are dependent iff one can find a nonzero constant K. As the coefficient of x^8 is $-K \sum_{i=1}^{9} x_i$, this implies $\sum_{i=1}^{9} x_i = 0$.

Suppose we have a minimal H-set for P_r, which contains some points of B, $t_1, t_2, \ldots, t_{m+1}$, and some intersection points $q_1, q_2, \ldots, q_h$ with $0 \leqslant h \leqslant l$. To check Theorem 4(2), we define for $k = h, \ldots, l-1$, the spaces T_k spanned by $\phi(t_1), \ldots, \phi(t_{m+1})$, $\phi(q_1), \ldots, \phi(q_k)$ and the varieties $Q_k = \{t \in \mathbb{R}^2; \phi(t) \in T_k\}$.

THEOREM 5. *Theorem* 4(2) *is fulfilled, i.e.,* $\{q_{h+1}, \ldots, q_l\}$ *does not form a support of* T_h, *iff* q_{k+1} *does not belong to* Q_k *for* $k = h, h+1, \ldots, l-1$.

Proof. Necessary condition. If $q_{k+1} \in Q_k$ for some $h \leqslant k < l$, one has

$$\phi(q_{k+1}) = \sum_{i=1}^{m+1} \lambda(t_i)\, \phi(t_i) + \sum_{j=1}^{k} \lambda(q_j)\, \phi(q_j)$$

or $\sum_{j=h+1}^{l} \lambda(q_j)\, \phi(q_j) \in T_h$ with $\lambda(q_{k+1}) = -1$ and $\lambda(q_j) = 0$ for $j > k+1$. Since $\lambda(q_{k+1})$ is nonzero, the set $\{q_{h+1}, \ldots, q_l\}$ is a support of T_h.

Sufficient condition. A support $\{q_{h+1}, \ldots, q_l\}$ of T_h implies

$$\sum_{j=h+1}^{l} \mu(q_j)\, \phi(q_j) = \sum_{i=1}^{m+1} \lambda(t_i)\, \phi(t_i) + \sum_{j=1}^{h} \lambda(q_j)\, \phi(q_j)$$

with at least one nonzero coefficient $\mu(q_j)$. If $k+1$ denotes the highest index of a nonzero $\mu(q_j)$, one obviously gets $q_{k+1} \in Q_k$. ∎

We illustrate the criterion by choosing $\mu = 1$, $\nu = 2$ with a couple of distinct intersection points q_1, q_2. For example, we start from a minimal H-set relative to $r = 3$, which is composed of ten points $t_i = (x_i, y_i)$, $i = 1, 2, \ldots, 10$, on the cubic C_3 defined by $y = x^3$, with $\sum_{j \neq i} x_j \neq 0$ for all i. If the H-set includes neither q_1 nor q_2, we get $Q_0 = C_3$ and q_1 must be located outside C_3. Hence, the second variety is $Q_1 = \mathbb{R}^2$ so that q_2 necessarily belongs to Q_1: the ten points $t_1, t_2, \ldots, t_{10}$ do not form a minimal H-set with respect to W. On the contrary, if $\sum_{i=1}^{9} x_i = 0$, the points $t_1, t_2, \ldots, t_9$ belong to C_3 and to another independent cubic Γ_3. One has $Q_0 = \{t_1, t_2, \ldots, t_9\}$ so that

TABLE II

μ	ν	$m+1$	Curves	Intersection Points
1	1	3	C_1	$q_1 \in C_1$
		4	C_1	$q_1 \notin C_1$
		5	C_2	$q_1 \in C_2$
		6	C_2	$q_1 \notin C_2$
		6	—	—
1	2	3	C_1	$q_1, q_2 \in C_2$
		4	C_1	$q_1 \in C_1, q_2 \notin C_1$
		5	C_1	$q_1, q_2 \notin C_1$
		6	C_2	$q_1, q_2 \in C_2$
		7	C_2	$q_1 \in C_2, q_2 \notin C_2$
		8	C_2	$q_1, q_2 \notin C_2$
		7	$C_3^{(1)} \cap C_3^{(2)}$	$q_1, q_2 \in C_3^{(1)} \cap C_3^{(2)}$
		8	$C_3^{(1)} \cap C_3^{(2)}$	$q_1 \in C_3^{(1)} \cap C_3^{(2)}, q_2 \notin C_3^{(1)} \cap C_3^{(2)}$
		9	$C_3^{(1)} \cap C_3^{(2)}$	$q_1, q_2 \notin C_3^{(1)} \cap C_3^{(2)}$
		8	C_3	$q_1, q_2 \in C_3$
		9	C_3	$q_1 \in C_3, q_2 \notin C_3$
		9	C_3	$q_1, q_2 \notin C_3$

q_1 may be everywhere in the plane. Any q_1 will define one given cubic from the pencil $aC_3 + b\Gamma_3$. Consequently, the nine points $t_1, t_2, \ldots, t_g$ build a minimal H-set for W provided q_2 does not lie on the unique cubic obtained by adding q_1 to the minimal-set.

Proceeding as above, we deduce from Table I the various minimal H-sets relative to rational functions characterized by $\mu = 1$, $\nu = 1$ and $\mu = 1$, $\nu = 2$. They are listed in Table II which a based on zero defect and distinct intersection points. The first three H-sets for $\mu = 1$, $\nu = 2$ are illustrated in Fig. 1. It must be emphasized that the intersection points may have complex or infinite values as shown in Figs. 2 and 3.

The foregoing developments hold true when some intersection points have multiplicity greater than one. An example is given in Fig. 4 for $\hat{g} = y/(x^2 - y)$ such that Theorem 2 imposes the conditions $w(0,0) = w_x(0,0) = 0$ for all $w \in W \subset P_3$. As only peculiarity, the points of the H-set may lie on curves which satisfy $w_x(0,0) = 0$ without passing through the origin. For instance, on the cubic $y = x^3 + 1$, W is the space of ninth degree polynomials

$$p(x) = \sum_{i=0}^{9} a_i x^i, \qquad a_8 = 0, \quad a_4 = a_1 + a_7. \tag{11}$$

As dim $W = 8$, any minimal H-set is, in general, composed of nine points except for special set of isolated points $x_1, x_2, \ldots, x_8$ which can be computed

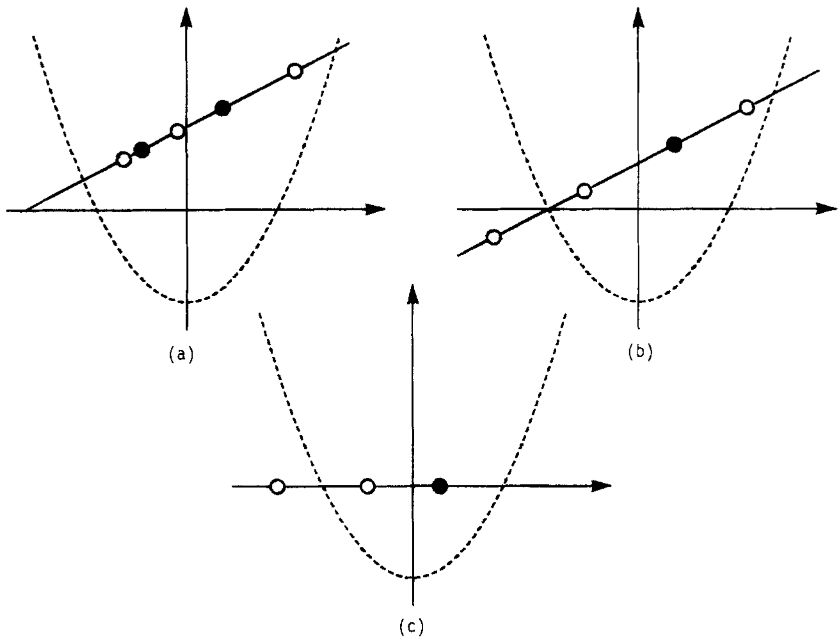

FIG. 1. Minimal H-Sets for $\hat{g} = y/(x^2 - y - 1)$ $[q_1 = (1, 0); q_2 = (-1, 0)]$.

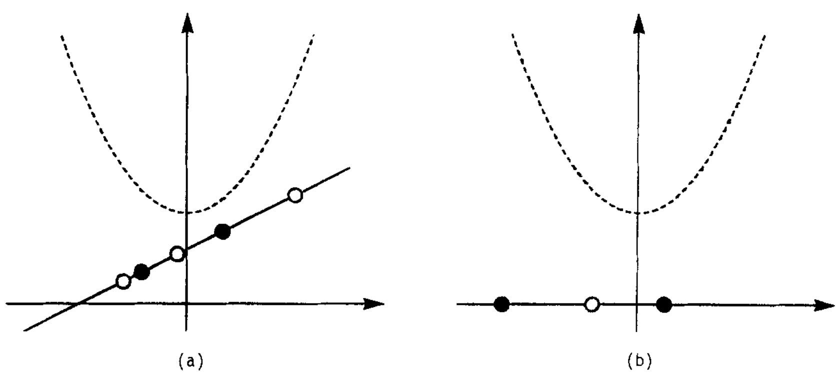

FIG. 2. Minimal H-Sets for $\hat{g} = y/(x^2 - y + 1)$ $[q_1 = (i, 0); q_2 = (-i, 0)]$.

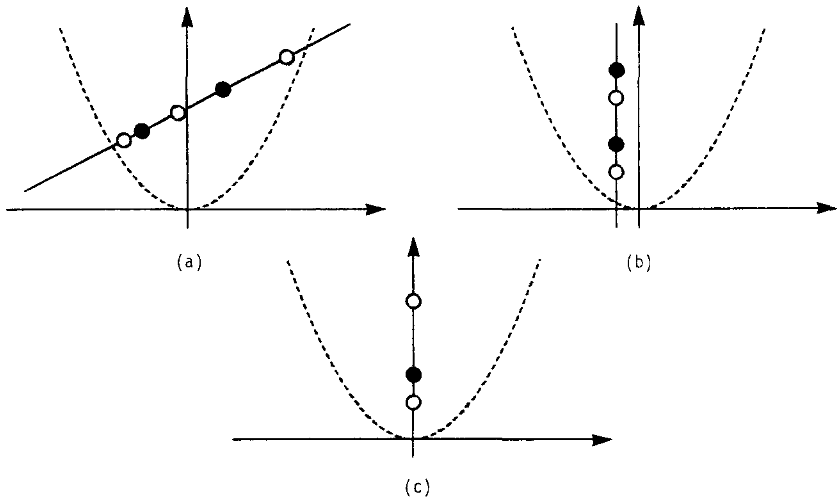

FIG. 3. Minimal H-Sets for $\hat{g} = x/(x^2 - y)$ $[q_1 = (X/Z,\ Y/Z)$ with $(X, Y, Z) = (0, 1, 0)$; $q_2 = (0, 0)]$.

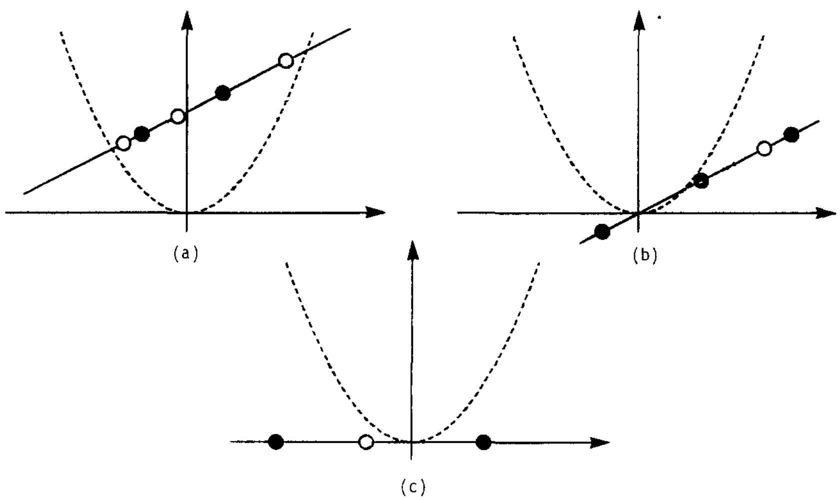

FIG. 4. Minimal H-Sets for $\hat{g} = y/(x^2 - y)$ $[q_1 = q_2 = (0, 0)]$.

by the previously mentioned method. The interpolation conditions $p(x_i)=0$, $i=1, 2,..., 8$, yield

$$p(x)=K\prod_{i=1}^{8}(x-x_i)(x-\alpha). \tag{12}$$

Hence, introducing $\prod_{i=1}^{8}(x-x_i)=\sum_{i=0}^{8} b_i x^i$ and identifying (12) with (11), we get

$$b_3-b_7b_4=b_0-b_7b_1+b_6-b_7^2.$$

For any set $A=\{x_1, x_2,..., x_7\}$, this relationship provides a second degree equation in x_8. For instance, $A=\{-3,-2,-1,0,1,2,3\}$ gives $x_8^{(1)}=(0.4)^{1/2}$ and $x_8^{(2)}=-(0.4)^{1/2}$. The set $A\cup\{x_8^{(1)}\}$ corresponds to the minimal H-set of eight isolated points reported in Table II but, in this case, there remains one more point defined by $x_8^{(2)}$ in the associated variety.

5. Singular Intersection Points

As shown at the end of Section 3, when the basic curves $\hat{u}=0$ and $\hat{v}=0$ have no contact at an intersection point q of order σ on $\hat{u}=0$ and of order τ on $\hat{v}=0$, the Noether exponent is $\rho=\sigma+\tau-1$. Whence, by taking the particular polynomials $u_i=v_i=0$ in the statement of Noether Theorem, we find the following well-known result [9]:

Theorem 6. *If $\hat{u}=0$ and $\hat{v}=0$ have no contact at their intersection point q, the polynomial w satisfies the Noether condition at q if q is a point of order $\sigma+\tau-1$ on $w=0$.*

We easily verify that this condition is only sufficient if q is singular on both curves. For instance, when $\sigma=\tau=2$, the Noether condition is satisfied iff there exist polynomials u and v such that q is a point of order three on $w=\hat{u}v+\hat{v}u$. This gives the four conditions $w=w_x=w_y=0$ and

$$\det\begin{pmatrix} w_{xx} & \hat{u}_{xx} & \hat{v}_{xx} \\ w_{xy} & \hat{u}_{xy} & \hat{v}_{xy} \\ w_{yy} & \hat{u}_{yy} & \hat{v}_{yy}\end{pmatrix}=0,$$

where all polynomials are evaluated at q. As we need necessary conditions for investigating H-sets, we can replace Theorem 6 by Proposition 3 which assumes $\sigma\leqslant\tau$.

Proposition 3. *The polynomial w satisfies the Noether condition at q only if q is a point of $w=0$, which is of order at least equal to σ.*

If minimal H-sets relative to bivariate polynomials include some singular point of order σ, the analysis performed in [16] still holds when the points of the H-set lie on one single curve. One has only to take into account the $\sigma(\sigma+1)/2$ conditions imposed by q. On the contrary, if the H-set belongs to several curves having no common part, the analysis fails because q produces σ^2 intersection points. As one has $\sigma(\sigma+1)/2=\sigma^2$ only for $\sigma=1$, the approach of [16] must be generalized. To this end, we consider minimal H-sets relative to P_r, which contain a fixed number M of singular points $q_1, q_2,..., q_M$ of order $\sigma_1, \sigma_2,..., \sigma_M$, in addition to some ordinary points $t_1, t_2,..., t_{m+1}$. We first assume that they lie on two curves $C_r^{(1)}$, $C_r^{(2)}$ without common part. Due to the Bezout theorem, we get

$$C_r^{(1)} \cap C_r^{(2)} = \{q_1,..., q_M, t_1,..., t_H\}, \tag{13}$$

where each t_i is supposed to be ordinary and $H \geqslant m+1$ is given by

$$H = r^2 - \sum_{i=1}^{M} \sigma_i^2. \tag{14}$$

The subset of (13) which yields independent conditions is $q_1,..., q_M$, $t_1,..., t_\rho$ with

$$\sum_{i=1}^{M} \sigma_i(\sigma_i+1)/2 + \rho = (r+1)(r+2)/2 - 2. \tag{15}$$

The remaining points $t_{\rho+1},..., t_H$ are said to be superabundant of superabundance $s=\sum_{i=1}^{M} \sigma_i(\sigma_i+1)/2 + H - (r+1)(r+2)/2$ or, by (14), $s=(r-1)(r-2)/2 - \sum_{i=1}^{M} \sigma_i(\sigma_i-1)/2$. For $M=0$, the set (13) is superabundant as soon as $r=3$ [16]. If there is some singular point, it is superabundant only for $r \geqslant 4$. For rational functions with $\mu=\nu=2$, such that $\hat{u}=0$ and $\hat{v}=0$ intersect in q_1 with $\sigma_1=\tau_1=2$, we get $s=2$ and $H=12$. Hence, the minimal H-set, which has superabundance one, is composed of $q_1, t_1, t_2,..., t_{11}$ and there is one more point t_{12} in the associated variety.

In order to find minimal H-sets belonging to several curves $C_r^{(i)}$ $(i=1,2,..., 2+b; b>0)$ with no common part, we shall determine from intersection set (13) related to the first curves, a superabundant subset $q_1,..., q_M$, $t_1,..., t_{\rho-b+s}$ in which $q_1,..., q_M, t_1,..., t_{\rho-b}$ provide independent conditions. By [7, p. 385; 10], the superabundance s is the number of independent curves of order $r-3$ which contain $q_1,..., q_M$ as points of order $\sigma_1-1,..., \sigma_M-1$, and which pass through $t_{\rho-b+s+1},..., t_H$. This gives a system of $\sum_{i=1}^{M} \sigma_i(\sigma_i-1)/2 + H - \rho + b - s$ or, by (14) and (15), $(r-1)(r-2)/2 + b - s$ linear equations in $(r-1)(r-2)/2$ unknowns. Since it has s independent solutions, its rank is equal to $(r-1)(r-2)/2 - s$ so that b equations are superabundant. Hence, the set $\{q_1,..., q_M, t_{\rho-b+s+1},..., t_H\}$, where each q_i is of order $\sigma_i - 1$, has superabundance b for degree $r-3$. The procedure described in [16] is still applicable. For instance, with $r=4$, $M=1$, $\sigma_1=2$, we start from a minimal superabundant set for degree one, which is the minimal H-set composed of three collinear points [16]. This gives $b=s=1$ and a minimal H-set $\{q_1, t_1,..., t_{10}\}$ for $W \subseteq P_4$, which belongs to three quartics. One may no longer add one more point on the straight line to get $b=2$, because it should be common to $C_4^{(1)}$ and $C_4^{(2)}$ in view of the Bezout theorem. In [16], we described minimal H-sets for P_4, which consist of twelve isolated points on four independent quartics. In fact, they are given by the intersection of a cubic with a quartic. For instance, on the parametrical cubic $y^2=x^3$, i.e., $x=z^2$, $y=z^3$, P_4 is the space of univariate polynomials $p(z)=\sum_{i=0}^{12} a_i z^i$ with $a_1=0$: the condition $\sum_{i=1}^{12} z_i^{-1}=0$ will thus produce twelve isolated points. If we now consider $\mu=\nu=2$, $\hat{u}=xy$, $\hat{v}=x^2+y^2$, on $y^2=x^3$, W is spanned by $z^4, z^5,..., z^{12}$ and we cannot find any set of isolated points.

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